THE LINEAR RESISTANCE BETWEEN PARALLEL CONDUCTING CYLINDERS IN A MEDIUM OF UNIFORM CONDUCTIVITY.

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It is the purpose of this paper to present formulas and tables for the computation of the linear resistances, conductances and capacities between parallel cylindrical conductors, or between a cylindrical conductor and a parallel indefinitely extending conducting plane. As is shown in the appended bibliography, the problem is by no means new; but the mathematical mode of presentation, and the arithmetical tabulation, here offered, are believed to be new. It is hoped that these will be useful to students of electrical engineering. Antihyperbolic functions are the natural vehicles of expression adapted to this problem.

Infinite Conducting Plane and Parallel Cylinder.

Linear Resistance.—Let a uniform conducting cylinder of radius

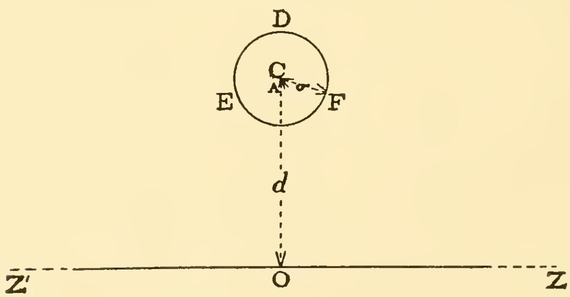

Fig. 1. Section of a conducting cylinder *DEF* parallel to the indefinitely extending conducting plane *Z'OZ*.

σ cm., shown in section at *DEF* in Fig. 1, be situated at an axial distance d cm. from a parallel indefinitely extending conducting plane $Z'OZ$. Let the space above the plane unoccupied by the cylinder be filled by an indefinitely extending medium of uniform resistivity ρ absohm-cm. Then the linear resistance between the plane and the cylinder, *i. e.*, the resistance of the medium between them, as comprised between a pair of infinite parallel planes perpendicular to the cylinder and 1 cm. apart, will be

$$r_p = \frac{\rho}{2\pi} \cosh^{-1}\left(\frac{d}{\sigma}\right) \quad \text{absohm-cms. or C.G.S. magnetic units of resistance in a linear cm.} \tag{1}$$

If the conducting surface EDF of the cylinder were unrolled into a flat conducting ribbon $2\pi\sigma$ cm. in breadth, and the ribbon were supported parallel to the plane $Z'OZ$ at a uniform distance $L = \sigma \cosh^{-1}(d/\sigma)$ cm. above it, as indicated in Fig. 2, with vertical insulating side walls, Ez' and Fz, to limit the flow of current through the medium to the parallel distribution shown; then the rectangular slab of medium $EFzz'$ of Fig. 2, would be the equivalent in electric resistance to the indefinitely extending plane and cylinder system of Fig. 1.

In Fig. 2 the depth, or distance across the slab, following the lines of current flow, is $L = \sigma \cosh^{-1}(d/\sigma)$ cm., and the

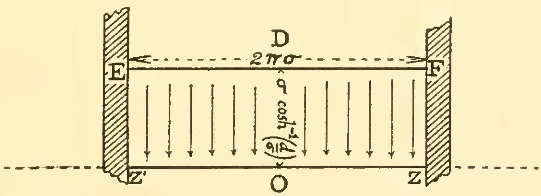

FIG. 2. Equivalent slab section corresponding to infinite plane and parallel cylinder of Fig. 1.

surface area of each face of the slab, per linear cm. of its length, is $S = 2\pi\sigma$ cm.^{2}/cm. so that the linear resistance of the whole is

$$r_p = \frac{L}{S}\rho = \rho\frac{\sigma \cosh^{-1}(d/\sigma)}{2\pi\sigma} = \frac{\rho}{2\pi}\cosh^{-1}\left(\frac{d}{\sigma}\right) \quad \text{absohm-cm.} \tag{2}$$

Since the linear resistance of the plane cylinder system of Fig. 1, or of the slab in Fig. 2, does not depend upon its absolute dimensions, the scale of linear dimensions in the diagram may be chosen such that $\sigma = 1$ unit, in which case the depth of the slab is $\cosh^{-1} d$ units and the breadth of the slab is 2π units.

The quantity Y defined by the relation

$$Y = \cosh^{-1}(d/\sigma) \qquad \text{numeric (3)}$$

may be called the *distance factor* of the plane-cylinder system; because the distance between electrodes in the equivalent slab of Fig. 2 is

$$L = Y\sigma \qquad \text{cm.}$$

When the radius σ of the cylinder is very small with respect to the distance d; so that d/σ is a large number, we have

$$Y = \log_e \frac{2d}{\sigma} \qquad \text{numeric (4)}$$

so that for such cylinders the linear resistance

$$r_p = \frac{\rho}{2\pi} Y = \frac{\rho}{2\pi} \log_e \frac{2d}{\sigma} \qquad \text{absohm-cm. (5)}$$

The accompanying table gives for successive values of d/σ in column I., the corresponding value of Y in column II. Column III. gives the resistance factor $Y/2\pi$ which, when multiplied by the resistivity ρ of the medium, gives the linear resistance of the plane-cylinder system considered.

Thus, if a conducting cylinder with a radius of 2 cm. is supported at an axial distance of 10 cm. from an infinite conducting plane, in a medium of resistivity $\rho = 3 \times 10^{10}$ absohm-cms., we have $d/\sigma = 5$. The table gives for this ratio the value of Y as 2.2924, and the value of the resistance factor $Y/2\pi = 0.3649$; so that the linear resistance of the system will be $3 \times 10^{10} \times 0.3649 = 1.0947 \times 10^{10}$ absohm-cms.; or 10.947 ohms in a linear cm.

Linear Conductance.—The linear conductance, or conductance per linear cm. of the plane-cylinder system will be by (1)

$$g_p = \frac{2\pi}{\rho \cosh^{-1}(d/\sigma)} = \frac{2\pi}{\rho Y} = \gamma \cdot \frac{2\pi}{Y} \qquad \text{abmhos per cm. (6)}$$

where γ is the uniform conductivity of the medium in abmhos per cm. The quantity $2\pi/Y$ may be called the *conductance-factor* of the plane-cylinder system. It appears in column V. of the table.

Thus, if a conducting cylinder of radius $\sigma = 0.5$ cm. be supported at an axial distance of $d = 7.5$ cm. from an infinite conducting plane, in a medium of conductivity $\gamma = 10^{-10}$ abmhos per cm., the ratio d/σ in column I. is 15, and the conductance factor for this ratio appears in column V. as 1.848. The linear conductance of the system is thus 1.848×10^{-10} abmhos per cm. The distance-factor of the system is given in column II. as 3.4001; so that the depth of the equivalent rectangular slab of medium is 1.700 cm., the breadth being 3.142 cm.

Linear Electrostatic Capacity.—The linear capacity c_p of a plane-cylinder system in a dielectric medium of specific inductive capacity κ, is numerically the same as the linear conductance of the same system in a medium of conductivity $\kappa/4\pi$ or resistivity $4\pi/\kappa$; so that, in C.G.S. electrostatic units:

$$c_p = \frac{\kappa}{2\cosh^{-1}(d/\sigma)} = \kappa \cdot \frac{1}{2Y} \quad \text{statfarads per cm.} \quad (7)$$

The values of the capacity factor $1/(2Y)$ appear in column VI. of the table for each selected value of d/σ.

Thus, a cylinder of radius $\sigma = 0.4$ cm. is supported at an axial distance of 1 cm. from an infinite conducting plane in a medium of $\kappa = 1$. Here $d/\sigma = 2.5$, and $1/(2Y) = 0.3192$. The linear capacity of the system is therefore 0.3192 statfarad per cm.

In order to convert the linear capacity c_p statfarads per cm. into microfarads per km., expressed by c_p', we have:

$$c_p' = \frac{c_p}{9} = \frac{\kappa}{9} \cdot \frac{1}{2Y} \quad \text{microfarads per km.} \quad (8)$$

Similarly, to express the linear capacity in microfarads per mile

$$c_p'' = \frac{c_p}{5.591} = \frac{\kappa}{5.591} \times \frac{1}{2Y} \quad \text{microfarads per mile} \quad (9)$$

That is, we must divide the capacity-factor of the table by 9 to obtain microfarads per km. or by 5.591 to obtain microfarads per mile.

Potential Distribution.

On the Median Line Beneath the Cylinder.—It is well known that the flow of electric current, and the distribution of potential, between the conducting cylinder and the plane, are such as might be produced by removing the conducting cylinder and substituting a conducting polar line at A, parallel to the plane. The point A lies on the line OC, and at a distance a from the plane defined by the relation

$$a = \sigma \sinh Y = \sqrt{d^2 - \sigma^2}. \qquad \text{cm. (10)}$$

The values of the polar ratio a/σ are given in the table in column VII. for each of the selected ratios d/σ up to $d/\sigma = 50$, beyond which the difference between a/σ and d/σ is less than 1 part in 5,000. For most practical purposes, it is, therefore, sufficient to regard the polar line as coinciding with the cylinder axis when the distance of that axis from the plane exceeds 50 radii.

In the steady state of flow, the potential at any point y_1 on the line OA (Fig. 3) distant y_1 cm. from O, will be

$$u_1 = I\frac{\rho}{\pi}\tanh^{-1}\left(\frac{y_1}{a}\right) \qquad \text{abvolts (11)}$$

where I is the current strength per linear cm. of the system in absamperes, the potential of the plane $Z'OZ$ being taken as numerically zero.

Similarly, the potential at any other point y_2 on the median line OY, below A, distant y_2 cm. from O, will be:

$$u_2 = I\frac{\rho}{\pi}\tanh^{-1}\left(\frac{y_2}{a}\right) \qquad \text{abvolts (12)}$$

Consequently, if the potential of the surface of the cylinder be u_1, and y_1 be the distance of the lowest point of the cylinder from the plane, the potential of any other point on the line OA between the cylinder and the plane, distant y_2 cm. from the latter, will be:

$$u_2 = u_1 \frac{\tanh^{-1}(y_2/a)}{\tanh^{-1}(y_1/a)} \qquad \text{abvolts (13)}$$

Potentials on the Median Line Above the Cylinder.—In the steady state of flow, the potential at any point y_3 on the median line OY, and distant y_3 cm. from O, above the polar point A, is:

$$u_3 = I\frac{\rho}{\pi}\coth^{-1}\left(\frac{y_3}{a}\right) \qquad \text{abvolts (14)}$$

where I and π have the same meanings as above, and the potential of the plane $Z'OZ$ is reckoned as zero.

Similarly, the potential at any other point y_4 on the median line OY, distant y_4 cm. from O, and above the polar point A, is:

$$u_4 = \frac{I\rho}{\pi}\coth^{-}\left(\frac{y_4}{a}\right) \qquad \text{abvolts (15)}$$

Consequently, if the potential of the surface of the cylinder be u_3, and y_3 be the distance of the highest point of the cylinder from the plane, the potential at any other point on the median line, above the cylinder, and distant y_4 cm. from the plane, will be:

$$u_4 = u_3\frac{\coth^{-1}(y_4/a)}{\coth^{-1}(y_3/a)} \qquad \text{abvolts (16)}$$

Potentials at Points Outside the Cylinder and off the Median Line.—If the point in the plane $Z'YZ$ at which the potential is required, lies off the median line OY, the potential may be expressed either:

(a) In terms of rectangular coördinates z and y of the point.

(b) In terms of the ratio of radii vectores to the point, from the polar point A, and from its image.

(a) *Potential in Terms of Rectangular Coördinates.*—Let P, Fig. 3, be the point whose potential is required, and whose rectangular coördinates are y and z, measured respectively along the median line OY, and the line OZ in the infinite conducting plane. Then u, the potential of P, is:

$$u = \frac{I\rho}{2\pi}\tanh^{-1}\left(\frac{2ay}{a^2+y^2+z^2}\right) \qquad \text{abvolts (17)}$$

where I, ρ and a have the values previously assigned, and the potential of the plane $Z'OZ$ is reckoned as zero. Eliminating $I\rho/\pi$ with the aid of (11), we have:

$$u = u_1\frac{\tanh^{-1}\left(\dfrac{2ay}{a^2+y^2+z^2}\right)}{2\tanh^{-1}(y_1/a)} \qquad \text{abvolts (18)}$$

u_1 is the potential of the conducting cylinder, upon the lowest point of which $y = y_1$ and $z = 0$. Thus, taking the point P in Fig. 3, defined by the coördinates $y = 1$ and $z = 2$, and referring the

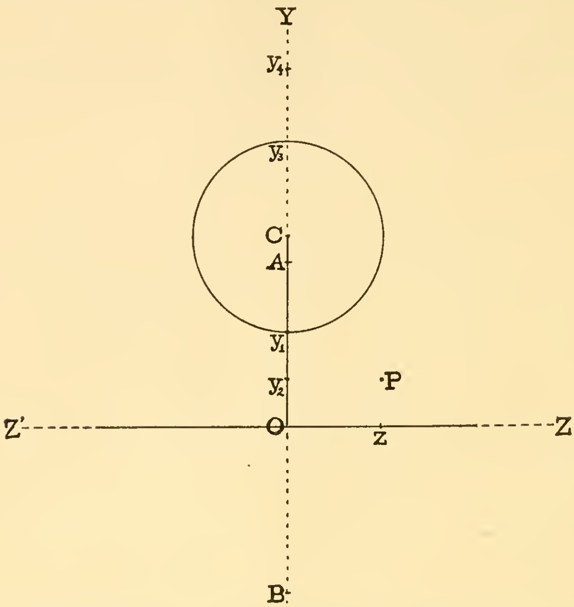

FIG. 3. Coördinates of a point at which the potential is required.

potential u of P to u_1, the potential of the surface of the cylinder, where $y_1 = 2$, $z = 0$, we have $a = 3.4642$ and

$$u = u_1 \frac{\tanh^{-1}(6.9284/17)}{2 \tanh^{-1}(2/3.4642)} = 0.3285 u_1.$$

Formula (18) may also be presented in the form:

$$u = u_1 \frac{\tanh^{-1}\left(\dfrac{2ay}{a^2 + y^2 + z^2}\right)}{\tanh^{-1}\left(\dfrac{2ay_1}{a^2 + y_1^2}\right)} \quad \text{abvolts} \quad (19)$$

(*b*) *Potential in Terms of Radii Vectores.*—A line parallel to the axis of the conducting cylinder, drawn through the point B, Fig. 3, on the median line OY and with the distance $OB = OA$, may be called the *image* of the polar line through OA. The point B, thus defined, may be called the *image polar point*. The points A and B, taken together, may be called the *polar points* of the diagram with respect to the infinite plane and cylinder.

Let P be any point in the plane of the diagram (Fig. 3). Then let r' and r be the lengths of a pair of radii vectores BP, AP, drawn from the polar points B, A, to P respectively. Let these distances $r'r$ be called the *polar distances* of the point P. Then the ratio m of these polar distances will be:

$$m = r'/r \qquad \text{numeric (20)}$$

This ratio may be called the *polar ratio*, for purposes of reference. The polar ratio will manifestly be a number greater than unity for all points in the diagram above the infinite conducting plane $Z'OZ$. It is a well known result that

$$u = \frac{I\rho}{2\pi} \log_e m \qquad \text{abvolts (21)}$$

If a point be selected on the surface of the cylinder, having a potential u_1 abvolts, and for convenience the lowest point of coördinates y_1 and $z = 0$, the polar distances of this point may be denoted by r_1' and r_1; while their ratio may be denoted by $m_1 = r_1'/r_1$. Consequently

$$u_1 = \frac{I\rho}{2\pi} \log_e m_1 \qquad \text{abvolts (22)}$$

and eliminating I, ρ and 2π between (21) (22), we have

$$u = u_1 \frac{\log_e m}{\log_e m_1} = u_1 \frac{\log_{10} m}{\log_{10} m_1} \qquad \text{abvolts (23)}$$

The potential of the infinite plane is here reckoned as zero. It may be observed that

$$m_1 = \frac{r_1'}{r_1} = \frac{a + d - \sigma}{r_1} = \frac{a + d}{\sigma} \qquad \text{numeric (24)}$$

When the cylinder radius is very small, compared with the axial distance d, $d = a$, and

$$m_1 = \frac{r_1'}{r_1} = \frac{2d}{\sigma} = \frac{D}{\sigma} \qquad \text{numeric (25)}$$

It follows from the preceding equations that the equipotential surfaces in an infinite plane-cylinder system are all cylinders having their axes situated on the median line. If u_1 be the potential of the conducting cylinder, and if we denote by Y_1 the value of the distance factor Y for this cylinder, according to formula (3), or to column II. of the table, then the distance factor Y of any cylindrical equipotential surface whose potential is u becomes

$$Y = Y_1 \frac{u}{u_1} \qquad \text{numeric (26)}$$

We have for any such cylinder the equations of condition:

$$Y = \cosh^{-1}(d/\sigma) = \sinh^{-1}(a/\sigma) = \tanh^{-1}(a/d) = \coth^{-1}(d/a)$$
$$= 2\tanh^{-1}(y/a) \qquad \text{numeric (27)}$$

whence d, the axial distance, or y coördinate, of the cylinder whose potential is u, will be along the median line OY:

$$d = \frac{a}{\tanh\left(Y_1 \frac{u}{u_1}\right)} \qquad \text{cm. (27)}$$

and the radius σ of this equipotential cylinder is:

$$\sigma = \frac{a}{\sinh\left(Y_1 \frac{u}{u_1}\right)} \qquad \text{cm. (28)}$$

The coördinate y of the lowest point of any such equipotential cylinder will be:

$$y = a\left(\frac{m - 1}{m + 1}\right) \qquad \text{cm. (29)}$$

$$= a \tanh\left(\frac{Y}{2}\right) = a \tanh\left(\frac{Y_1}{2}\frac{u}{u_1}\right) \qquad \text{cm. (30)}$$

so that

$$u = u_1 \frac{\tanh^{-1}\left(\frac{m - 1}{m + 1}\right)}{\tanh^{-1}(y_1/a)} \qquad \text{abvolts (31)}$$

an expression for the potential of a point in the medium in terms of its polar ratio m, and the distance y_1 of the conducting cylinder from the plane.

The current density δ at any point whose polar distances are r and r' will be perpendicular to the equipotential cylinder passing through the point and will be equal to

$$\delta = I\frac{\rho}{\pi} \cdot \frac{a}{rr'} \quad \text{absamperes per cm.}^2 \qquad (31a)$$

The preceding formulas for potential distribution have been developed with reference to a conducting medium between the infinite plane and cylinder. They are, however, applicable to the case of a dielectric medium, if the electric flux ϕ replace the electric current I, and the dielectric constant κ be substituted for γ or $1/\rho$. No substitution will be needed in formulas (13), (16), (18), (19) and (23) to (31), inclusive, which apply either to an insulating or to a conducting medium.

Two Equal and Parallel Conducting Cylinders.

If, instead of an infinite conducting plane and a parallel conducting cylinder, as in Figs. 1 and 3, we have two indefinitely long parallel conducting cylinders of equal diameter, as in Fig. 4, at an interaxial distance CC' of D cm., then each cylinder may be regarded as forming an independent plane-cylinder system with a fictitious infinite conducting midplane $Z'OZ$, axially distant $d = D/2$ cm. from each. This midplane will be perpendicular to the central line CC'. The double-cylinder system will have two polar lines equidistant from the system center O, and represented in Fig. 4 by the polar points AA'. The potential of the midplane $Z'OZ$ will be midway between the potentials of the two cylinders; so that if these have equal and opposite potentials, the potential of the midplane will be zero. All of the preceding formulas for plane-cylinder systems may, therefore, be applied, in duplicate, to the double-cylinder system of Fig. 4.

Linear Resistance of Double Cylinder Systems.—The linear resistance from either cylinder to the midplane is given in formula

(1). Consequently, the linear resistance of the double cylinder system of Fig. 4 is

$$r_{00} = \frac{\rho}{\pi} \cosh^{-1}(d/\sigma) = \frac{\rho}{\pi} Y \qquad \text{absohm-cms.} \quad (32)$$

where $d = D/2$. The resistance factor of the system is thus Y/π, or double that given in column III. of the table.

Thus, if the two cylinders, each of radius $\sigma = 2$ cm. separated

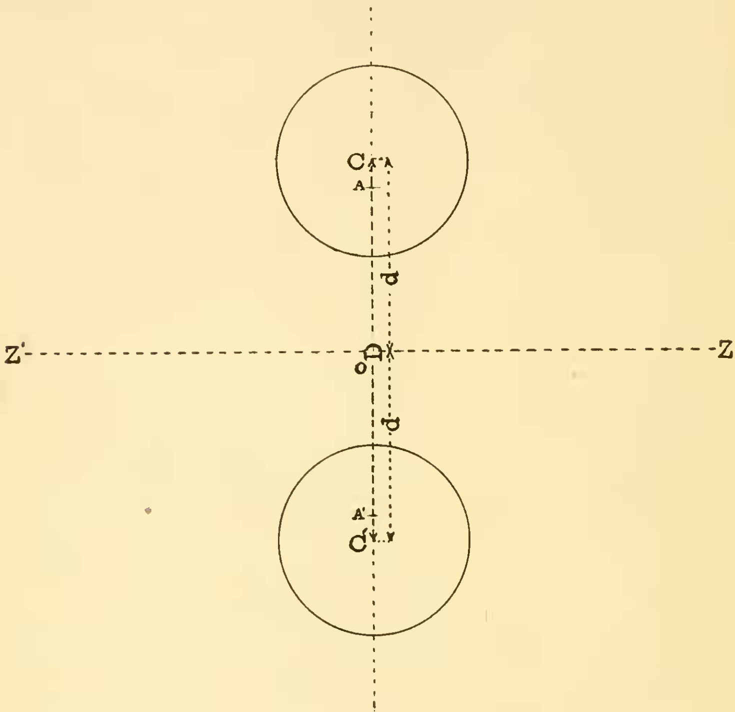

FIG. 4. Two equal and parallel conducting cylinders at interaxial distance of D cm.

by an interaxial distance $D = 8$ cm. in a medium of resistivity $\rho = 5 \times 10^{10}$ absohm-cms. we have $d = 4$, and $d/\sigma = 2$.

$Y = \cosh^{-1} 2 = 1.317$, and the linear resistance is

$$r_{00} = \frac{5 \times 10^{10}}{3.1416} \times 1.317 = 2.096 \times 10^{10} \qquad \text{absohm-cms.}$$

Linear Conductance of Double-Cylinder Systems.—The linear conductance of a double cylinder system will be half that of a plane-cylinder system of equal d/σ; so that:

$$g_{00} = \frac{\pi}{\rho \cosh^{-1}(d/\sigma)} = \frac{\pi}{\rho Y} = \frac{\gamma\pi}{Y} \quad \text{abmhos per cm.} \quad (33)$$

where γ is the conductivity of the medium. The conductance-factor of the double-cylinder system is therefore half of that given in column V. of the table.

Linear Electrostatic Capacity of Double-Cylinder Systems.—The linear capacity C_{00} of a double-cylinder system in a dielectric medium of specific capacity κ is half the capacity of a plane-cylinder system of equal d/σ; so that:

$$c_{00} = \frac{\kappa}{4 \cosh^{-1}(d/\sigma)} = \kappa \cdot \frac{1}{4Y} \quad \text{statfarads per loop cm.} \quad (34)$$

The linear capacity of each cylinder to the zero-potential plane, or the capacity of the system per cylinder-cm., is given by formula (7). The capacity factors of a double-cylinder system of given d/σ are thus half of the values given in column VI. of the table; but the capacity factors of the system per "wire" cm. to zero potential midplane are those recorded in column VI.

At interaxial distances large with respect to the cylinder-radii, $Y = \log_\epsilon D/\sigma$, and we obtain the well known formula

$$c_{00} = \frac{\kappa}{4 \log_\epsilon(D/\sigma)} \quad \text{statfarads per cm.} \quad (35)$$

The linear capacity of a double-cylinder system expressed in microfarads per km. is

$$c_{00}' = \frac{c_{00}}{9} = \frac{\kappa}{9} \cdot \frac{1}{4Y} \quad \text{microfarads per cm.} \quad (36)$$

Similarly,

$$c_{00}'' = \frac{c_{00}}{5.591} = \frac{\kappa}{5.591} \times \frac{1}{4Y} \quad \text{microfarads per mile} \quad (37)$$

Potential Distribution in Double Cylinder System.—All of the formulas (10) to (31) inclusive referring to the potential distribution in a plane-cylinder system apply immediately to a double-cylinder system, after the latter has been analyzed into two associated plane-cylinder systems.

Two Unequal Parallel Conducting Cylinders.

Let two parallel conducting cylinders, with their axes at C_1C_2, Fig. 5, have unequal radii σ_1 and σ_2 cm., and be separated by an interaxial distance D cm. If the radii were equal, the midplane $z'z$ would be the plane of zero potential, when the potentials of the cylinders are equal and opposite. The zero-potential plane is, how-

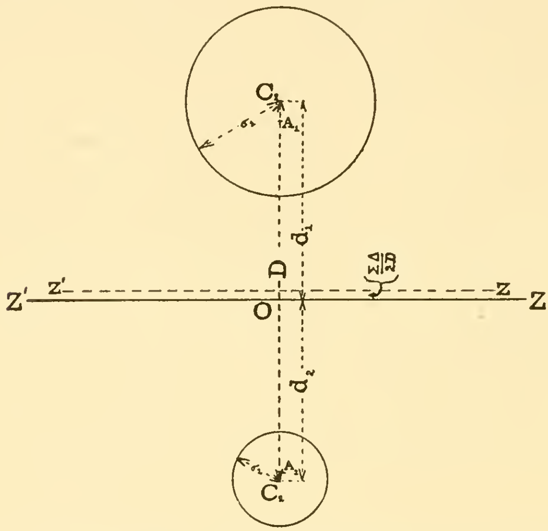

Fig. 5. Two unequal parallel conducting cylinders at interaxial distance of D cm. showing the displacement of the zero-potential plane.

ever, displaced from the larger towards the smaller cylinder through a distance of $\Sigma\Delta/2D$ cm.; so that:

$$\left.\begin{aligned} d_1 &= \frac{D}{2} + \frac{\Sigma\Delta}{2D} \quad \text{cm.} \\ d_2 &= \frac{D}{2} - \frac{\Sigma\Delta}{2D} \quad \text{cm.} \end{aligned}\right\} \qquad (38)$$

where $\Sigma = \sigma_1 + \sigma_2$ is the sum and $\Delta = \sigma_1 - \sigma_2$ is the difference of the cylinder radii.

After having established the position of the zero-potential plane $Z'OZ$, the linear resistance between the cylinders may be found by using formula (1) on each side of the plane and adding the two parts. The linear conductance will then be the reciprocal of this result.

The linear capacity of each cylinder to zero-potential plane is to be found by formula (7). The linear capacity per loop cm. may be found from the linear resistance per loop cm. by the formula:

$$c_{00} = \frac{\kappa}{2(Y_1 + Y_2)} \quad \text{statfarads per cm.} \quad (39)$$

For example, if two conducting cylinders of radii $\sigma_1 = 2$ and $\sigma_1 = 1$ cm., respectively, are separated in air by an interaxial distance of 8 cm., the zero-potential plane is displaced through a distance of $\frac{3}{16}$ cm., so that $d_1 = 4\frac{3}{16}$, $d_2 = 3\frac{13}{16}$ cm. The ratio d_1/σ_1 is thus 2.094, and d_2/σ_2 is 3.815. The distance factor Y_1 is 1.37, and Y_2 is 2.014. The linear capacity of C_1 is 0.365 statfarads per cm. and of C_2 0.248 statfarads per cm., each to zero-potential plane. The linear capacity of the pair by (39) is 0.1477 statfarad per loop cm.

The potential distribution in the unequal cylinder system may be obtained as easily as when the cylinders are equal, since the polar points A_1A_2, Fig. 4, lie at equal distances from the zero-potential plane $Z'OZ$.

Excentric Cylinders.

Let the two parallel very thin conducting cylinders be hollow, with radii σ_1 and σ_2. Let one be placed excentrically within the other, as shown in Fig. 6, at an interaxial distance D. Let the line C_1C_2 joining their centers be prolonged as indicated in the figure. The infinite zero-potential plane will perpendicularly intersect this line at an inferred distance of $\Sigma\Delta/2D$ cm. from the middle point of D; so that:

$$d_1 = \frac{\Sigma\Delta}{2D} + \frac{D}{2} \quad \text{cm.} \quad (40)$$

and

$$d_2 = \frac{\Sigma\Delta}{2D} - \frac{D}{2} \quad \text{cm.} \quad (41)$$

The linear resistance between the cylinders can now be determined by finding the linear resistance of each to the infinite conducting plane by formula (1) and then taking the difference between these linear resistances.

Thus, let $\sigma_1 = 4$ cm., $\sigma_2 = 2$ cm., $D = 1$ cm. Then $\Sigma = 6$, $\Delta = 2$, and $d_1 = 6.5$ cm., $d_2 = 5.5$ cm.

The resistance factor for d_2 by the table is 0.2657.

The resistance factor for d_1 by the table is 0.1697.

The resistance factor between d_2 and d_1 = 0.0960.

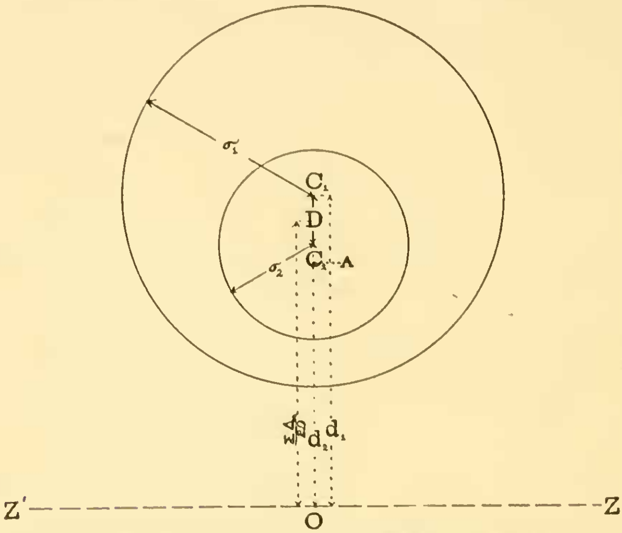

FIG. 6. Two parallel excentric cylinders, one enclosing the other, and the inferred common zero-potential plane.

which multiplied by the resistivity of the medium gives the linear resistance between the cylinders.

Through the use of formulas (40) and (41) all cases of excentric cylinders may be computed by reduction to the equivalent pair of plane-cylinder systems.

Graphical Construction of Equipotential and Stream Lines in a Plane-Cylinder System.

To draw the equipotential and stream lines of a plane-cylinder system, when the polar distance OA or distance a of the polar axis from the parallel plane is known, draw zOK, Fig. 7, to represent the plane and on the median line OY, perpendicular to zOK mark off, to scale, the polar distance $a = OA$. Then to locate any equipotential circle of radius $\sigma = OE'$, mark off with center O, a distance $d = OC = AE'$. With center C and the required radius σ, describe the equipotential circle FEB. The distance factor Y for this circle will be expressed by

$$Y = 2 \tanh^{-1}\left(\frac{y_1}{a}\right) \qquad \text{numeric} \quad (42)$$

where y_1 is the distance OF or the y coördinate of the lowest point

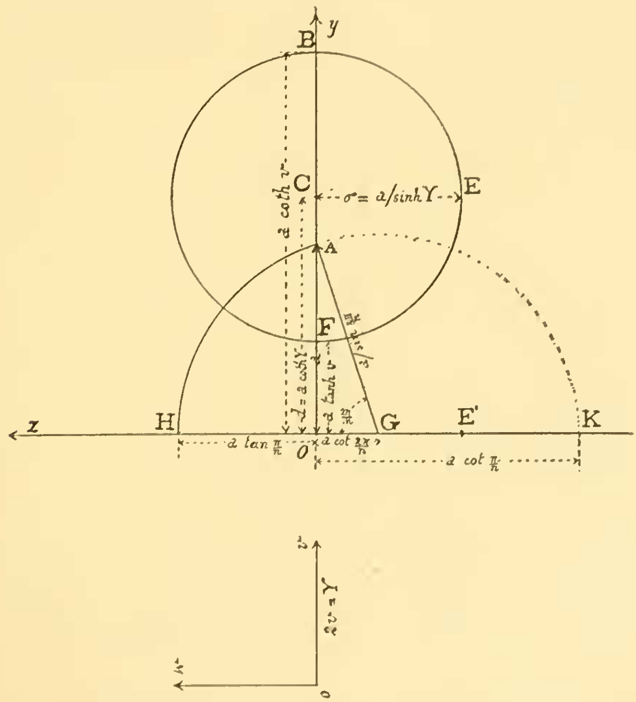

FIG. 7. Diagram for graphic construction of equipotential and stream lines.

on the circle. The potential of the circle with reference to the plane will be

$$u = \frac{I\rho}{2\pi} Y \qquad \text{abvolts} \quad (43)$$

To draw a stream line which shall include with the median line OA the nth part of all the linear flux in the system, mark off on OK a distance $OG = a \cot 2\pi/n$; so that the angle OGA will contain $2\pi/n$ radians. Then with center G and radius GA, describe the circular arc AH, which is the required stream-line.

It may be observed that if we draw two coördinate axes ov ow in the vw plane, the function $\tanh (v + w\sqrt{-1})$ will correspond on the yz plane to the required loci, magnified by a. The locus of this function, when v is given successive constant values and w alone varies, is a series of equipotential circles, while when w is successively assigned constant values and v alone varies, the loci of successive stream-lines are produced. If w is expressed in terms of π as π/n and $2v = Y$, we have

$$OF = a \tanh v = d - \sigma \qquad \text{cm.} \quad (44)$$

$$OB = a \coth v = d + \sigma \qquad \text{cm.} \quad (45)$$

$$CE = a/\sinh Y = \sigma \qquad \text{cm.} \quad (46)$$

$$OC = a \coth Y = d \qquad \text{cm.} \quad (47)$$

also

$$OH = a \tan \pi/n \qquad \text{cm.} \quad (48)$$

$$OK = a \cot \pi/n \qquad \text{cm.} \quad (49)$$

$$GA = a/\sin (2\pi/n) \qquad \text{cm.} \quad (50)$$

$$OG = a \cot 2\pi/n \qquad \text{cm.} \quad (51)$$

Fig. 8 presents the graphical construction of the function $\tanh (v + w\sqrt{-1})$ carried from the vw plane to the yz plane, over the limits $v = -1$ to $v = +1$ and $w = -\pi/2$ to $w = +\pi/2$. The points marked on the vw plane have their corresponding points marked on the yz plane. Thus the point p defined by $v = 1.0$, $w = \pi/2$ on the vw plane is represented by the point p defined by $y = 1.313$, $z = 0$, on the yz plane, or $\tanh (1 + \pi/2 \cdot \sqrt{-1}) = 1.313$. Corresponding areas on the two planes are shaded alike. It follows from the formulas already discussed that linear resistances, conductances and capacities are the same between corresponding conducting surfaces in the two diagrams. Thus, the linear resistance of the double-cylinder system $pqrs$—$tuvx$ is equal to the linear resistance of the rectangular slab system with $pqrs$ as one electrode and $tuvx$

as the other; *i. e.*, $2/\pi$ absohm-cm. Moreover, the linear resistance of any curvilinear element, such as between *qr* on one cylinder, and *uv* on the other, in the *yz* system, is equal to the linear resistance between the parallel electrodes *qr* and *uv* on the rectilinear *vw* system ($10/\pi$ absohm-cms. with unit resistivity).

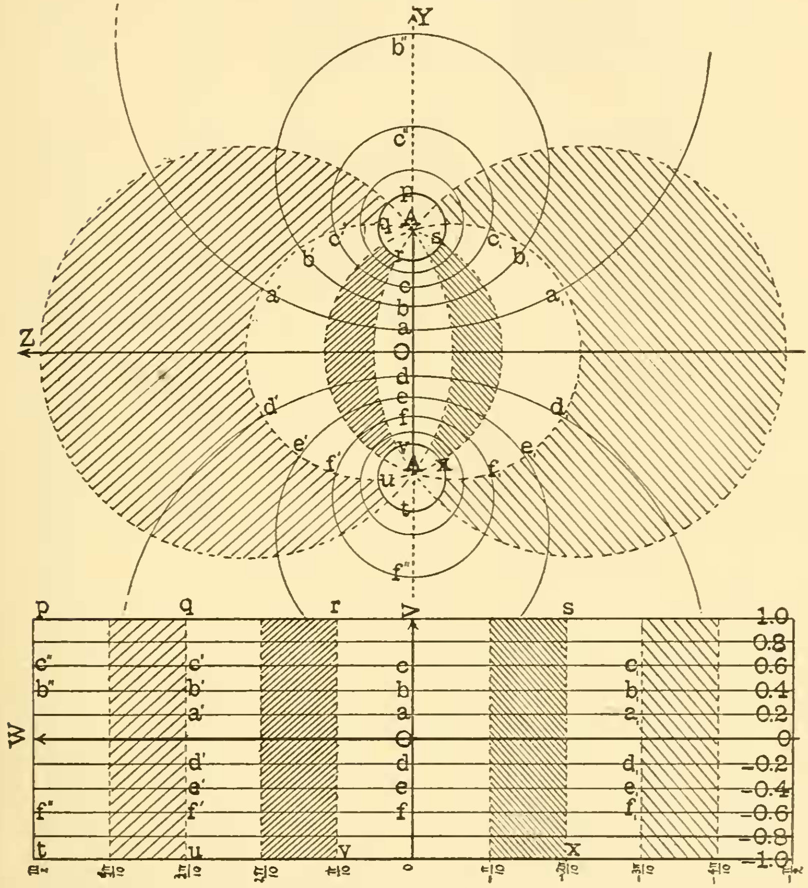

FIG. 8. Graphical comparison of $(v + w\sqrt{-1})$ and of tanh $(v + w\sqrt{-1})$.

In Fig. 8, $a = OA = 1$; but it is easy to see that the proposition of equal linear resistances, conductances and capacities between corresponding conductors in the double-cylinder and corresponding rectangular slab systems, is independent of the magnification in the diagram.

I	II Distance Factor	III Resistance Factor	IV	V Conductance Factor	VI Capacity Factor	VII sinh Y
d/σ	$Y=\cosh^{-1}\left(\frac{d}{\sigma}\right)$	$Y/2\pi$	$1/Y$	$2\pi/Y$	$1/(2Y)$	$\frac{a}{\sigma}=\sqrt{\left(\frac{d}{\sigma}\right)^2-1}$
1.01	0.1413	0.0225	7.0787	44.47	3.5393	0.1418
1.05	0.3149	0.0501	3.1756	19.95	1.5878	0.3202
1.1	0.4435	0.0706	2.2548	14.16	1.1274	0.4582
1.2	0.6224	0.0991	1.6067	10.095	0.8034	0.6633
1.3	0.7564	0.1204	1.3221	8.307	0.6611	0.8307
1.4	0.8670	0.1380	1.1534	7.246	0.5767	0.9798
1.5	0.9622	0.1531	1.0393	6.531	0.5197	1.1180
1.6	1.0470	0.1666	0.9551	6.002	0.4776	1.2490
1.7	1.1232	0.1788	0.8901	5.594	0.4451	1.3748
1.8	1.1929	0.1899	0.8383	5.267	0.4191	1.4967
1.9	1.2569	0.2001	0.7956	4.999	0.3978	1.6156
2.0	1.3170	0.2096	0.7593	4.771	0.3797	1.7321
2.1	1.3729	0.2185	0.7284	4.576	0.3642	1.8466
2.2	1.4255	0.2296	0.7015	4.407	0.3508	1.9596
2.3	1.4750	0.2348	0.6780	4.259	0.3390	2.0712
2.4	1.5216	0.2422	0.6572	4.129	0.3286	2.1817
2.5	1.5668	0.2494	0.6383	4.010	0.3192	2.2913
2.6	1.6096	0.2562	0.6214	3.903	0.3107	2.4000
2.7	1.6502	0.2626	0.6059	3.807	0.3030	2.5080
2.8	1.6886	0.2688	0.5922	3.721	0.2961	2.6153
2.9	1.7267	0.2748	0.5791	3.639	0.2896	2.7221
3.0	1.7627	0.2806	0.5673	3.564	0.2837	2.8284
3.1	1.7975	0.2861	0.5563	3.495	0.2782	2.9343
3.2	1.8309	0.2914	0.5462	3.432	0.2731	3.0397
3.3	1.8633	0.2966	0.5367	3.372	0.2684	3.1448
3.4	1.8946	0.3015	0.5278	3.317	0.2639	3.2496
3.5	1.9248	0.3063	0.5195	3.264	0.2598	3.3541
3.6	1.9542	0.3110	0.5117	3.215	0.2559	3.4583
3.7	1.9827	0.3156	0.5044	3.169	0.2522	3.5623
3.8	2.0104	0.3200	0.4974	3.126	0.2487	3.6661
3.9	2.0373	0.3242	0.4909	3.084	0.2454	3.7696
4.0	2.0634	0.3284	0.4846	3.045	0.2423	3.8730
4.1	2.0889	0.3325	0.4787	3.008	0.2394	3.9762
4.2	2.1137	0.3364	0.4731	2.973	0.2366	4.0792
4.3	2.1380	0.3402	0.4677	2.939	0.2339	4.1821
4.4	2.1616	0.3440	0.4626	2.907	0.2313	4.2849
4.5	2.1846	0.3477	0.4577	2.876	0.2289	4.3875
4.6	2.2072	0.3513	0.4531	2.847	0.2265	4.4900
4.7	2.2292	0.3548	0.4486	2.819	0.2243	4.5924
4.8	2.2507	0.3582	0.4443	2.792	0.2221	4.6947
4.9	2.2718	0.3616	0.4402	2.766	0.2201	4.7969
5.0	2.2924	0.3649	0.4362	2.741	0.2181	4.8990
5.1	2.3126	0.3681	0.4324	2.717	0.2162	5.0010
5.2	2.3324	0.3712	0.4287	2.694	0.2144	5.1029
5.3	2.3514	0.3743	0.4253	2.672	0.2127	5.2048

I d/σ	II Distance Factor $Y=\cosh^{-1}\left(\frac{d}{\sigma}\right)$	III Resistance Factor $Y/2\pi$	IV $1/Y$	V Conductance Factor $2\pi/Y$	VI Capacity Factor $1/(2Y)$	VII $\sinh Y$ $\frac{a}{\sigma}=\sqrt{\left(\frac{d}{\sigma}\right)^2-1}$
5.4	2.3709	0.3773	0.4218	2.650	0.2109	5.3066
5.5	2.3895	0.3803	0.4185	2.630	0.2093	5.4083
5.6	2.4078	0.3832	0.4153	2.610	0.2077	5.5100
5.7	2.4258	0.3861	0.4122	2.590	0.2061	5.6116
5.8	2.4435	0.3889	0.4093	2.571	0.2047	5.7131
5.9	2.4608	0.3917	0.4064	2.553	0.2032	5.8146
6.0	2.4779	0.3944	0.4036	2.536	0.2018	5.9161
6.5	2.5590	0.4073	0.3908	2.455	0.1954	6.4226
7.0	2.6339	0.4192	0.3797	2.386	0.1898	6.9282
7.5	2.7036	0.4303	0.3699	2.324	0.1849	7.4330
8.0	2.7687	0.4407	0.3612	2.270	0.1806	7.9373
8.5	2.8297	0.4503	0.3539	2.224	0.1770	8.4410
9.0	2.8873	0.4596	0.3463	2.176	0.1732	8.9443
9.5	2.9417	0.4682	0.3399	2.136	0.1700	9.4472
10.0	2.9932	0.4764	0.3341	2.099	0.1670	9.9499
11	3.0890	0.4916	0.3237	2.034	0.1619	10.9545
12	3.1763	0.5055	0.3148	1.978	0.1574	11.9583
13	3.2566	0.5183	0.3071	1.930	0.1536	12.9615
14	3.3309	0.5301	0.3002	1.887	0.1501	13.964
15	3.4001	0.5411	0.2941	1.848	0.1471	14.967
16	3.4648	0.5514	0.2886	1.814	0.1443	15.969
17	3.5255	0.5611	0.2837	1.782	0.1418	16.971
18	3.5827	0.5702	0.2791	1.754	0.1396	17.972
19	3.6369	0.5788	0.2750	1.728	0.1375	18.974
20	3.6882	0.5870	0.2712	1.704	0.1356	19.975
21	3.7371	0.5948	0.2676	1.681	0.1338	20.976
22	3.7837	0.6022	0.2643	1.661	0.1321	21.977
23	3.8282	0.6093	0.2612	1.641	0.1306	22.978
24	3.8708	0.6161	0.2584	1.623	0.1292	23.979
25	3.9116	0.6226	0.2557	1.606	0.1278	24.980
26	3.9509	0.6287	0.2531	1.590	0.1266	25.981
27	3.9887	0.6348	0.2507	1.575	0.1254	26.981
28	4.0250	0.6406	0.2485	1.561	0.1243	27.982
29	4.0604	0.6462	0.2463	1.548	0.1232	28.983
30	4.0941	0.6516	0.2443	1.535	0.1221	29.983
32	4.1590	0.6619	0.2404	1.511	0.1202	31.984
34	4.2193	0.6715	0.2370	1.489	0.1185	33.985
36	4.2765	0.6806	0.2338	1.469	0.1169	35.986
38	4.3306	0.6892	0.2309	1.451	0.1155	37.987
40	4.3819	0.6972	0.2282	1.434	0.1141	39.987
42	4.4307	0.7051	0.2257	1.418	0.1129	41.988
44	4.4772	0.7126	0.2234	1.403	0.1117	43.989
46	4.5217	0.7196	0.2212	1.390	0.1106	45.989
48	4.5642	0.7264	0.2191	1.377	0.1096	47.990
50	4.6051	0.7329	0.2172	1.364	0.1086	49.990

I d/σ	II Distance Factor $Y = \cosh^{-1}\left(\frac{d}{\sigma}\right)$	III Resistance Factor $Y/2\pi$	IV $1/Y$	V Conductance Factor $2\pi/Y$	VI Capacity Factor $1/(2Y)$	VII sinh Y $\frac{a}{\sigma} = \sqrt{\left(\frac{d}{\sigma}\right)^2 - 1}$
52	4.6443	0.7392	0.2153	1.353	0.1077	52
54	4.6821	0.7452	0.2136	1.342	0.1068	54
56	4.7184	0.7509	0.2119	1.332	0.1060	56
58	4.7535	0.7565	0.2104	1.322	0.1052	58
60	4.7874	0.7619	0.2088	1.312	0.1044	60
65	4.8676	0.7747	0.2054	1.291	0.1027	65
70	4.9416	0.7864	0.2024	1.272	0.1012	70
75	5.0106	0.7975	0.1996	1.254	0.0998	75
80	5.0751	0.8077	0.1970	1.238	0.0985	80
85	5.1358	0.8173	0.1947	1.224	0.0974	85
90	5.1930	0.8264	0.1926	1.210	0.0963	90
95	5.2470	0.8350	0.1906	1.198	0.0953	95
100	5.2983	0.8433	0.18874	1.1859	0.09437	100
110	5.3936	0.8585	0.18540	1.1648	0.09270	110
120	5.4806	0.8723	0.18246	1.1464	0.09123	120
130	5.5607	0.8852	0.17983	1.1298	0.08992	130
140	5.6348	0.8969	0.17747	1.1150	0.08874	140
150	5.7038	0.9078	0.17532	1.1016	0.08766	150
160	5.7683	0.9180	0.17336	1.0892	0.08668	160
170	5.8290	0.9278	0.17156	1.0778	0.08578	170
180	5.8861	0.9369	0.16989	1.0674	0.08495	180
190	5.9402	0.9456	0.16834	1.0577	0.08417	190
200	5.9915	0.9536	0.16690	1.0486	0.08345	200
220	6.0868	0.9688	0.16429	1.0322	0.08215	220
240	6.1738	0.9827	0.16197	1.0176	0.08099	240
260	6.2538	0.9954	0.15990	1.0047	0.07995	260
280	6.3279	1.0071	0.15803	0.9930	0.07902	280
300	6.3969	1.0180	0.15633	0.9822	0.07817	300
320	6.4615	1.0283	0.15476	0.9725	0.07738	320
340	6.5221	1.0381	0.15322	0.9634	0.07666	340
360	6.5793	1.0471	0.15199	0.9550	0.07600	360
380	6.6333	1.0557	0.15075	0.9473	0.07538	380
400	6.6846	1.0639	0.14960	0.9400	0.07480	400
420	6.7334	1.0716	0.14851	0.9332	0.07426	420
440	6.7799	1.0790	0.14749	0.9268	0.07375	440
460	6.8244	1.0862	0.14653	0.9207	0.07327	460
480	6.8669	1.0929	0.14563	0.9151	0.07282	480
500	6.9078	1.0993	0.14476	0.9096	0.07238	500
550	7.0031	1.1146	0.14279	0.8972	0.07140	550
600	7.0901	1.1284	0.14104	0.8862	0.07052	600
650	7.1701	1.1411	0.13947	0.8764	0.06974	650
700	7.2442	1.1530	0.13804	0.8674	0.06902	700
750	7.3132	1.1640	0.13674	0.8591	0.06837	750
800	7.3778	1.1741	0.13554	0.8518	0.06777	800
850	7.4384	1.1838	0.13444	0.8449	0.06722	850

I d/σ	II Distance Factor $Y = \cosh^{-1}\left(\frac{d}{\sigma}\right)$	III Resistance Factor $Y/2\pi$	IV $1/Y$	V Conductance Factor $2\pi/Y$	VI Capacity Factor $1/(2Y)$	VII sinh Y $\frac{a}{\sigma}=\sqrt{\left(\frac{d}{\sigma}\right)^2-1}$
900	7.4955	1.1930	0.13341	0.8383	0.06671	900
950	7.5496	1.2016	0.13246	0.8323	0.06623	950
1000	7.6009	1.2097	0.13156	0.8266	0.06578	1000
1100	7.6962	1.2249	0.12993	0.8165	0.06497	1100
1200	7.7832	1.2387	0.12848	0.8074	0.06424	1200
1300	7.8633	1.2515	0.12717	0.7990	0.06359	1300
1400	7.9374	1.2632	0.12599	0.7916	0.06300	1400
1500	8.0064	1.2742	0.12490	0.7848	0.06245	1500
1600	8.0709	1.2845	0.12390	0.7786	0.06195	1600
1700	8.1315	1.2940	0.12298	0.7728	0.06149	1700
1800	8.1887	1.3032	0.12212	0.7674	0.06106	1800
1900	8.2428	1.3118	0.12132	0.7624	0.06066	1900
2000	8.2941	1.3200	0.12056	0.7575	0.06028	2000
2100	8.3428	1.3278	0.11986	0.7532	0.05993	2100
2200	8.3894	1.3351	0.11920	0.7490	0.05960	2200
2300	8.4338	1.3423	0.11857	0.7451	0.05929	2300
2400	8.4764	1.3490	0.11798	0.7414	0.05899	2400
2500	8.5172	1.3555	0.11741	0.7378	0.05871	2500
2600	8.5564	1.3618	0.11687	0.7344	0.05844	2600
2700	8.5942	1.3678	0.11636	0.7312	0.05818	2700
2800	8.6305	1.3735	0.11587	0.7280	0.05794	2800
2900	8.6656	1.3791	0.11540	0.7251	0.05770	2900
3000	8.6995	1.3845	0.11495	0.7224	0.05748	3000
3100	8.7323	1.3898	0.11452	0.7196	0.05726	3100
3200	8.7641	1.3949	0.11410	0.7170	0.05705	3200
3300	8.7948	1.3996	0.11370	0.7144	0.05685	3300
3400	8.8247	1.4045	0.11332	0.7121	0.05666	3400
3500	8.8537	1.4090	0.11295	0.7098	0.05648	3500
3600	8.8818	1.4135	0.11259	0.7075	0.05630	3600
3700	8.9092	1.4180	0.11224	0.7053	0.05612	3700
3800	8.9359	1.4220	0.11191	0.7032	0.05596	3800
3900	8.9619	1.4262	0.11158	0.7012	0.05579	3900
4000	8.9872	1.4302	0.11127	0.6992	0.05564	4000
4100	9.0118	1.4342	0.11097	0.6973	0.05549	4100
4200	9.0360	1.4381	0.11067	0.6954	0.05534	4200
4300	9.0595	1.4419	0.11038	0.6936	0.05519	4300
4400	9.0825	1.4456	0.11010	0.6918	0.05505	4400
4500	9.1050	1.4491	0.10983	0.6902	0.05492	4500
4600	9.1270	1.4526	0.10957	0.6885	0.05479	4600
4700	9.1485	1.4560	0.10931	0.6869	0.05466	4700
4800	9.1695	1.4593	0.10906	0.6853	0.05453	4800
4900	9.1901	1.4627	0.10881	0.6838	0.05441	4900
5000	9.2103	1.4659	0.10857	0.6822	0.05429	5000

Notation.

a = polar distance or distance of polar axis from parallel plane in a plane-cylinder system, cm.
c_p = linear capacity of plane-cylinder system, statfarads/cm.
c_p' = linear capacity of plane-cylinder system, microfarads/km.
c_p'' = linear capacity of plane-cylinder system, microfarads/mile
c_{00} = linear capacity of double-cylinder system, statfarads/cm.
c_{00}' = linear capacity of double-cylinder system, microfarads/km.
c_{00}'' = linear capacity of double-cylinder system, microfarads/mile
d = distance of cylinder axis from plane, cm.
$d_1 d_2$ = distances of cylinder axes from plane in double-cylinder system with unequal cylinders, cm.
D = $2d$ or interaxial distance between two cylinders in a double cylinder system, cm.
$\Delta = \sigma_1 - \sigma_2$ = difference in radii of two cylinders, cm.
δ = current density at a point in the medium, absamperes/cm.^{2}.
g_p = linear conductance of plane-cylinder system, abmho/cm.
g_{00} = linear conductance of double-cylinder system, abmho/cm.
κ = specific inductive capacity of medium,
γ = conductivity of medium, abmho/cm.
I = linear current in a system, absamperes/cm.
L = length of flux paths in rectangular slab, cm.
m = r'/r, polar ratio, or ratio of vector lengths from poles to a point in the medium, numeric
$1/n$ = a fractional part of the total linear flux, limited by a stream line.
$\pi = 3.14159 \cdots$.
$r_1 r'$ = polar distances or vector lengths from poles to a point.
r_p = linear resistance of a plane-cylinder system absohm/cm.
r_{00} = linear resistance of a double-cylinder system, absohm/cm.
ϕ = linear electric flux in a system, statmaxwells/cm.
ρ = resistivity of medium, absohm-cm.
S = linear surface area of a conducting slab, cm.^{2}/cm.
$\Sigma = \sigma_1 + \sigma_2$ = sum of radii of two unequal cylinders, cm.
σ = radius of a cylinder, cm.
u = potential of a cylinder, abvolts or statvolts
vw = rectangular coördinates of points in a plane, cm.
Y = distance factor of a system $= \cosh^{-1}(d/\sigma)$, numeric
yz = rectangular coördinates of points in a plane, cm.
$y_1 y_2$ = y-coördinates of points on median line below a cylinder, cm.
$y_3 y_4$ = y-coördinates of points on median line above a cylinder, cm.

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